The “Dirac delta function” strategy for solving the Poisson equation

Contents

1 Introduction

The Poisson equation is

$$\Delta u = f \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given continuous function. Here Δ is the Laplace operator defined by

$$\Delta u = \text{div } \nabla u$$

for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$. Equation (1) is one of the great equations in all of science.

My goal in these notes is to give an intuitive explanation of the “Dirac delta function” strategy for constructing a solution to the Poisson equation (1). Physics textbooks often explain this technique in a way that is so non-rigorous that it is in fact just nonsense. If you then go to a mathematician for help, they might tell you that in order to explain this technique correctly we must develop the whole formalism of distributions (generalized functions), which could be a textbook in itself. But it is possible to understand this topic very clearly, at an intuitive (non-rigorous) level, both without speaking nonsense and without developing the heavy machinery of distributions. We only need to phrase the physics explanation a little more carefully.

The "Dirac delta function" is supposed to have properties that no function $f : \mathbb{R}^n \to \mathbb{R}$ could possibly have. A mathematician could give the delta function a precise definition as a "generalized function", but in that approach the intuition is lost. The key to understanding the delta function intuitively is to think in terms of an "approximate delta function", which is a perfectly ordinary, smooth function that has a spike of volume 1 near the origin and is zero elsewhere. After obtaining an approximate solution to the Poisson equation by working with an "approximate delta function", we can take a limit (imagine the spike grows sharper and sharper) to obtain an exact solution. The calculations in these notes are very similar to calculations that appear in the textbook Introduction to Electrodynamics by Griffiths. The only difference is that I have made this intuitive line of reasoning explicit.

As a bonus, we'll show how to compute the divergence and the curl of the electric field (in electrostatics) and of the magnetic field (in magnetostatics).

2 Some facts from vector calculus

In this section we'll review some useful facts from vector calculus.

2.1 Some notation

This section specifies some notation that we'll use in these notes.

- If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $x \in \mathbb{R}^n$, then $F'(x)$ denotes the $m \times n$ Jacobian matrix of F at x. In the special case that $F : \mathbb{R}^n \to \mathbb{R}$, $F'(x)$ is a $1 \times n$ matrix (a row vector), and the gradient of F at x is the column vector
$$\nabla F(x) = F'(x)^T.$$
- To say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is "smooth" means (in these notes) that $f \in C^\infty(\mathbb{R}^n)$. In other words, to say that f is smooth means that all partial derivatives of f, of all orders, exist at each point $x \in \mathbb{R}^n$.
- To say that a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ has "compact support" means that f is zero outside of some bounded subset of $\mathbb{R}^n$. In other words, it means that there exists a number $R > 0$ such that $f(x) = 0$ for all $x \in \mathbb{R}^n$ such that $\|x\| > R$. For simplicity, I'll assume that the function f in equation (1) has compact support.
- $v \cdot w$ denotes the dot product of vectors $v, w \in \mathbb{R}^n$.
- $v \times w$ denotes the cross product of vectors $v, w \in \mathbb{R}^3$.
- If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable vector field on $\mathbb{R}^n$, then $\nabla \cdot f$ denotes $\operatorname{div} f$, the divergence of f.
- If $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable vector field on $\mathbb{R}^3$, then $\nabla \times f$ denotes $\operatorname{curl} f$, the curl of f.

2.2 A product rule for the divergence

Suppose that U is an open subset of $\mathbb{R}^n$ and that $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}^n$ are differentiable functions. Then
$$\operatorname{div}(uv) = \nabla u \cdot v + u \operatorname{div} v. \tag{2}$$

2.3 An integration by parts formula

Now suppose that U is a bounded, open subset of $\mathbb{R}^n$ and that ∂U, the boundary of U, is a smooth surface (if $n = 3$) or smooth manifold (if $n > 3$). Integrating both sides of equation (2) over U and applying the divergence theorem, we obtain

$$\begin{aligned}\int_{\partial U} uv \cdot dA &= \int_U \operatorname{div}(uv)\, dx \\ &= \int_U \nabla u \cdot v \, dx + \int_U u \operatorname{div} v \, dx.\end{aligned}$$

It follows that

$$\int_U \nabla u \cdot v \, dx = -\int_U u \operatorname{div} v \, dx + \int_{\partial U} uv \cdot dA. \tag{3}$$

Equation (3) can be interpreted as stating that the adjoint of the gradient operator is the negative divergence operator, in a setting where the boundary term vanishes. (Aesthetically, mathematicians should have never introduced the divergence operator — it would have been more beautiful to focus attention on $-\operatorname{div}$, which could be called the "convergence".)

2.4 Differentiating under the integral sign

There are various theorems to the effect that it is allowable to "differentiate under the integral sign". Here is the basic idea behind these theorems. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function with compact support and $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \int_{\mathbb{R}} F(x, y)\, dy.$$

for all $x \in \mathbb{R}$. Then

$$\begin{aligned}f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} &= \lim_{\Delta x \to 0} \int_{\mathbb{R}} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x}\, dy \\ &= \int_{\mathbb{R}} \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x}\, dy \\ &= \int_{\mathbb{R}} \frac{\partial F(x, y)}{\partial x}\, dy.\end{aligned}$$

So, in conclusion,

$$f'(x) = \int_{\mathbb{R}} \frac{\partial F(x, y)}{\partial x}\, dy. \tag{4}$$

This is an example of a "differentiation under the integral" formula. The difficult part of rigorously proving this type of formula is to justify the step where the limit is pulled inside the integral. Certainly, that step is quite plausible and you would at least hope it is valid. Rigorous proofs can be found in real analysis textbooks. For example, the dominated convergence theorem and the monotone convergence theorem each guarantee that "the limit of the integral is the integral of the limit", under certain mild conditions.

We'll need the particular results recorded below.

2.4.1 Bringing the divergence and curl inside the integral

Suppose that $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a smooth function with compact support. I'll write $F(x, y)$ instead of the more proper $F(\begin{bmatrix} x \\ y \end{bmatrix})$ when $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$f(x) = \int_{\mathbb{R}^m} F(x, y)\, dy$$

for all $x \in \mathbb{R}^n$. Then

$$(\text{div } f)(x) = \int_{\mathbb{R}^m} \text{div } F(x,y)\, dy. \tag{5}$$

In the expression on the right, the divergence is computed with respect to x, just as the partial derivative in equation (4) is with respect to x. Equation (5) can be thought of as a continuous version of the fact that "the divergence of the sum is the sum of the divergences".

When $n = 3$, we also have

$$(\nabla \times f)(x) = \int_{\mathbb{R}^m} (\nabla \times F)(x,y)\, dy. \tag{6}$$

In the expression on the right, the curl is computed with respect to x, just as the partial derivative in equation (4) is with respect to x. Equation (6) is a continuous version of the fact that "the curl of the sum is the sum of the curls".

2.5 The divergence and curl of $v \times w$

In section 5 on magnetostatics, we'll need the following identities. Suppose that $v : \mathbb{R}^3 \to \mathbb{R}^3$ and $w : \mathbb{R}^3 \to \mathbb{R}^3$ are differentiable vector fields on $\mathbb{R}^3$. The divergence of $v \times w$ is given by

$$\nabla \cdot (v \times w) = w \cdot (\nabla \times v) - v \cdot (\nabla \times w). \tag{7}$$

The curl of $v \times w$ is given by

$$\nabla \times (v \times w) = v'w - w'v + (\nabla \cdot w)v - (\nabla \cdot v)w. \tag{8}$$

Is the meaning of $v'w$ clear? If $x \in \mathbb{R}^3$ then $v'(x)$ is a 3×3 matrix, and so $(v'w)(x) = v'(x)w(x) \in \mathbb{R}^3$. We are multiplying a 3×3 matrix by a 3×1 column vector.

3 Solving the Poisson equation

Now we are ready to describe a particular strategy for constructing a solution to the Poisson equation (1). I'll attempt to explain this technique intuitively rather than rigorously. For simplicity, I'll take $n = 3$, and I'll assume that f is continuous with compact support.

3.1 The "delta function" strategy

Let $\epsilon > 0$ be a very small positive number, and let $\delta_\epsilon : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with the following two properties:

1. $\delta_\epsilon(x) = 0$ for all x such that $\|x\| \geq \epsilon$.
2. $\int_{\mathbb{R}^3} \delta_\epsilon(x)\, dx = 1$.

In other words, δ_ϵ has a sharp spike near the origin and is zero elsewhere. (I might later refer to δ_ϵ as a "spike function" for that reason.)

Suppose that we are somehow able to find a smooth function $\Phi_\epsilon : \mathbb{R}^3 \to \mathbb{R}$ which satisfies the Poisson equation equation (1) in the special case that $f = \delta_\epsilon$:

$$\Delta \Phi_\epsilon = \delta_\epsilon.$$

This function Φ_ϵ is a **building block** from which we are able to construct solutions to equation (1) for many other choices of f. For example, if $y \in \mathbb{R}^3$, then

$$\Delta\Big(\Phi_\epsilon(x-y)\Big) = \delta_\epsilon(x-y).$$

(We are differentiating with respect to x.) So we are now able to solve the Poisson equation when the function f in equation (1) is any shifted version of δ_ϵ. That is progress!

Next, what if the right hand side in equation (1) is a linear combination of shifted versions of δ_ϵ? We can handle that case too. If $y_1, \dots, y_M \in \mathbb{R}^3$ and $c_1, \dots, c_M \in \mathbb{R}$, then

$$\Delta \left(\sum_{i=1}^{M} c_i \Phi_\epsilon(x - y_i) \right) = \sum_{i=1}^{M} c_i \delta_\epsilon(x - y_i).$$

Continuing this line of thought, we can pass from a sum to an integral and note that

$$\Delta \left(\int_{\mathbb{R}^3} c(y) \Phi_\epsilon(x-y)\, dy \right) = \underbrace{\int_{\mathbb{R}^3} c(y) \delta_\epsilon(x-y)\, dy}_{\bigstar} \tag{9}$$

for any function $c : \mathbb{R}^3 \to \mathbb{R}$ which is continuous and has compact support. Equation (9) follows from the "differentiation under the integral sign" rule. So, as long as the function f in equation (1) has the form ($\bigstar$), we can find a solution to the Poisson equation.

The punch line is that *any* function f that is continuous with compact support can be written in the form ($\bigstar$), to a good approximation. If $x \in \mathbb{R}^3$ then

$$f(x) \approx \int_{\mathbb{R}^3} f(y) \delta_\epsilon(x-y)\, dy. \tag{10}$$

Equation (10) expresses f as a sum of shifted spike function. To see why equation (10) is true, let $B_\epsilon(x)$ be the closed ball of radius ϵ centered at x. Since f is continuous and ϵ is very small, it seems reasonable to assume that f is approximately constant on $B_\epsilon(x)$, so that $f(y) \approx f(x)$ for all $y \in B_\epsilon(x)$. It follows that

$$\int_{\mathbb{R}^3} f(y)\delta_\epsilon(x-y)\, dy = \int_{B_\epsilon(x)} f(y)\delta_\epsilon(x-y)\, dy \approx \int_{B_\epsilon(x)} f(x)\delta_\epsilon(x-y)\, dy = f(x) \underbrace{\int_{B_\epsilon(x)} \delta_\epsilon(x-y)\, dy}_{1} = f(x).$$

In conclusion, equation (9) with $c = f$ combines with equation (10) to reveal that the function

$$u_\epsilon(x) = \int_{\mathbb{R}^3} f(y) \Phi_\epsilon(x-y)\, dy$$

satisfies the Poisson equation (1), to a good approximation:

$$\Delta u_\epsilon \approx f. \tag{11}$$

Finally, how can we find an exact solution to equation (1)? Suppose that for each $\epsilon > 0$, no matter how tiny, we can construct a smooth function $\Phi_\epsilon : \mathbb{R}^3 \to \mathbb{R}$ whose Laplacian $\delta_\epsilon = \Delta \Phi_\epsilon$ satisfies the two properties listed above. (Namely, $\delta_\epsilon(x) = 0$ when $\|x\| \geq \epsilon$ and $\int_{\mathbb{R}^3} \delta_\epsilon(x)\, dx = 1$.) It seems plausible that the error in the approximation (11) can be made as small as we like by choosing ϵ to be sufficiently small. If there exists a function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\Phi_\epsilon \to \Phi$ as $\epsilon \to 0$, then it seems plausible that the function

$$u(x) = \lim_{\epsilon \to 0} u_\epsilon(x) = \int_{\mathbb{R}^3} f(y) \Phi(x-y)\, dy \tag{12}$$

will satisfy the Poisson equation (1) exactly. So, the function u defined in equation (15) is our solution to the Poisson equation (1).

Comments:

- Non-rigorous treatments of this topic, in physics textbooks or otherwise, often introduce the "Dirac delta function" δ which is said to be zero everywhere except at the origin, and yet also satisfies $\int_{\mathbb{R}^3} \delta(x)\, dx = 1$. Typically the author acknowledges that no such function actually exists. But then how do we make sense of what we are doing? While it is true that this description of the Dirac delta function can be made rigorous by developing the machinery of "generalized functions" (distributions), that is a lot of work, and it is surely not what physicists or mathematicians had in mind when they first invented the Dirac delta function. To understand the idea intuitively, without speaking nonsense, I believe one must think in terms of "approximate delta functions" such as the functions δ_ϵ described above. I suspect this is what everyone who understands the delta function intuitively has in mind, but it is not always made explicit.

- The same strategy presented here also works for solving $Lu = f$, where L is any linear differential operator.

- If you'd like to see a rigorous proof that the function $u(x) = \int_{\mathbb{R}^3} f(y)\Phi(x-y)\, dx$ satisfies $\Delta u = f$, one good place to look is the textbook Partial Differential Equations by Evans.

3.2 A fundamental solution for the Laplace operator in three dimensions

So far we have not yet shown how to construct the functions Φ_ϵ that we need in order to execute on the strategy described above. We will construct suitable functions Φ_ϵ in this section. This will enable us to write down a solution to equation (1).

Let $\Phi : \mathbb{R}^3 - \{0\} \to \mathbb{R}$ be the function defined by

$$\Phi(x) = \frac{-1}{4\pi}\|x\|^{-1} \qquad \text{for all } x \in \mathbb{R}^3,\ x \neq 0. \tag{13}$$

This famous function is called the "fundamental solution for the Laplace operator" in three dimensions. The functions Φ_ϵ will be constructed by smoothing out Φ near the origin. Admittedly I am pulling this function Φ out of thin air — how to discover this function is a separate question that we can ponder at another time. In the rest of this section, we will calculate the Laplacian of Φ, and then show that the functions Φ_ϵ have the desired properties.

Note that

$$\Phi(x) = \frac{-1}{4\pi} h(x)^{-1}$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$h(x) = \|x\| \qquad \text{for all } x \in \mathbb{R}^3.$$

For $x \neq 0$ the derivative of h is

$$h'(x) = \frac{x^T}{\|x\|}.$$

By the chain rule,

$$\Phi'(x) = \frac{1}{4\pi} h(x)^{-2} h'(x) = \frac{1}{4\pi}\frac{x^T}{\|x\|^3},$$

and so

$$\nabla\Phi(x) = \frac{1}{4\pi}\frac{x}{\|x\|^3}.$$

A straightforward calculation shows that

$$\Delta\Phi(x) = (\operatorname{div}\nabla\Phi)(x) = 0 \qquad \text{for all } x \in \mathbb{R}^3,\ x \neq 0. \tag{14}$$

This is one of those situations where showing the details will make the calculation seem more complicated than it would if you were to simply do it on paper. Nevertheless, one way to establish (14) is to use the product rule (2), with $u(x) = h(x)^{-3}$ and $v(x) = x$. If $x \neq 0$ then

$$\nabla u(x) = -3h(x)^{-4}\nabla h(x) = -3\frac{1}{\|x\|^4}\frac{x}{\|x\|} = -3\frac{x}{\|x\|^5}$$

and $(\operatorname{div} v)(x) = 3$. By the product rule (2) we have

$$\operatorname{div}(uv)(x) = \nabla u(x) \cdot v(x) + u(x)(\operatorname{div} v)(x) = -3\frac{x}{\|x\|^5} \cdot x + 3\|x\|^{-3} = -3\frac{\|x\|^2}{\|x\|^5} + 3\|x\|^{-3} = 0.$$

3.3 Constructing the functions Φ_ϵ

Now let's construct functions Φ_ϵ. Let $\epsilon > 0$, and let B_ϵ be the closed ball of radius ϵ centered at the origin. Let $\Phi_\epsilon : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function which agrees with Φ on $\mathbb{R}^3 - B_\epsilon$. In other words, Φ_ϵ is a smooth function which satisfies

$$\Phi_\epsilon(x) = \Phi(x) \qquad \text{for all } x \in \mathbb{R}^3 \text{ such that } \|x\| \geq \epsilon.$$

We see immediately that the function $\delta_\epsilon = \Delta\Phi_\epsilon$ is zero when $\|x\| \geq \epsilon$. We can also see that

$$\begin{aligned}
\int_{\mathbb{R}^3} \delta_\epsilon(x)\,dx &= \int_{B_\epsilon} \operatorname{div} \nabla\Phi_\epsilon(x)\,dx \\
&= \int_{\partial B_\epsilon} \nabla\Phi_\epsilon \cdot dA \\
&= \int_{\partial B_\epsilon} \frac{1}{4\pi}\frac{x}{\|x\|^3} \cdot dA \\
&= \frac{1}{4\pi\epsilon^2} \underbrace{\int_{\partial B_\epsilon} \frac{x}{\|x\|} \cdot dA}_{\text{Area of } \partial B_\epsilon} \\
&= 1.
\end{aligned}$$

This shows that δ_ϵ satisfies the two properties of a "spike function" listed in section 3.1. By construction we have that $\Phi_\epsilon \to \Phi$ as $\epsilon \to 0$. So, according to the strategy presented in section 3.1, the function

$$\boxed{u(x) = \int_{\mathbb{R}^3} f(y)\Phi(x-y)\,dy} \tag{15}$$

with Φ given by equation (13) is a solution to the Poisson equation (1).

Warning: It might be tempting to compute the Laplacian of the function u in equation (15) as follows:

$$\begin{aligned}
\Delta u(x) &= \int_{\mathbb{R}^3} f(y) \underbrace{\Delta\Phi(x-y)}_{0 \text{ when } y \neq x}\,dy \qquad (??) \\
&= 0.
\end{aligned}$$

The above calculation is incorrect, though, because moving the Laplace operator inside the integral in the line marked (??) is an error. There is no theorem which justifies that step. This example shows that we must be careful when "differentiating under the integral sign".

4 The divergence and curl of an electric field (in electrostatics)

Imagine a physical situation where some stationary electric charge is spread out over some region in space. For example, I've read that before lightning strikes, the top of a storm cloud acquires an excess of positive charge and the bottom of the storm cloud acquires an excess of negative charge. What force does this distribution of charge exert on a separate collection of charge that is concentrated very near a point x? (This separate concentration of charge might be called a "particle" located at x.) To answer this question, Coulomb's model of electric forces introduces a function $\rho : \mathbb{R}^3 \to \mathbb{R}$ which we think of as telling us the charge density at each location in space for the first collection of charge (which is smeared out throughout a cloud, for example) as well as a number q which we think of as the amount of charge that is concentrated near the point x. (So q is the charge of the "particle" located at the point x.) Coulomb's model asserts that the force on the charged particle is $F = qE(x)$, where E is the vector field on $\mathbb{R}^3$ defined by

$$E(x) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(y) \frac{(x-y)}{\|x-y\|^3} \, dy = \frac{1}{\epsilon_0} \int_{\mathbb{R}^3} \rho(y) \nabla\Phi(x-y) \, dy.$$

Here Φ is the function defined in equation (13). The scalar ϵ_0 is a parameter in Coulomb's model.

The goal of this section is to compute the divergence and curl of E. Our calculation will again be non-rigorous, but hopefully straightforward.

4.1 The divergence of E

Let $\epsilon > 0$ be a very small positive number, and define Φ_ϵ as in section 3.1. (So $\Phi_\epsilon : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function which agrees with Φ outside of B_ϵ, the closed ball of radius ϵ centered at the origin.) We have seen previously that $\delta_\epsilon = \Delta\Phi_\epsilon$ is a "spike function". Note that if $x \in \mathbb{R}^3$ then

$$E(x) \approx \frac{1}{\epsilon_0} \int_{\mathbb{R}^3} \rho(y) \nabla\Phi_\epsilon(x-y) \, dy \tag{16}$$

and so

$$\begin{aligned}
(\operatorname{div} E)(x) &\approx \frac{1}{\epsilon_0} \int_{\mathbb{R}^3} \rho(y) \operatorname{div} \nabla\Phi_\epsilon(x-y) \, dy \\
&= \frac{1}{\epsilon_0} \int_{\mathbb{R}^3} \rho(y) \, \Delta\Phi_\epsilon(x-y) \, dy \\
&= \frac{1}{\epsilon_0} \int_{\mathbb{R}^3} \rho(y) \, \delta_\epsilon(x-y) \, dy \\
&\approx \frac{1}{\epsilon_0} \rho(x).
\end{aligned}$$

In the final step, we used equation (10).

It seems plausible that we could make the approximation as close as we like by choosing ϵ to be sufficiently small. It follows that

$$\operatorname{div} E = \frac{1}{\epsilon_0} \rho.$$

This famous equation is called Gauss's law (in differential form). It is one of Maxwell's equations.

4.2 The curl of E

Let $x \in \mathbb{R}^3$. Starting from equation (16), we find that

$$(\nabla \times E)(x) \approx \frac{1}{\epsilon_0} \int_{\mathbb{R}^3} \rho(y) \underbrace{(\nabla \times \nabla\Phi_\epsilon)}_{\substack{\text{curl of} \\ \text{gradient} \\ \text{is } 0}}(x-y) \, dy = 0.$$

It seems plausible that we could make the approximation as close as we like by choosing ϵ to be sufficiently small. It follows that

$$\nabla \times E = 0.$$

This is also one of Maxwell's equations.

5 The divergence and curl of the magnetic field (in magnetostatics)

Imagine that a steady current of electric charge is flowing through space. For example, electric current could be flowing through a thick wire. Empirically, a charged particle that moves near this current will feel a force that seems to somehow be due to the current. (At least, if the current is turned off, the particle will feel no such force.) To mathematically model the force exerted on the charged particle at a particular moment, we introduce the following mathematical quantities:

- A vector $v \in \mathbb{R}^3$, the instantaneous velocity of the charged particle.
- A scalar q that we think of as telling us how much charge the particle has.
- A vector field J on $\mathbb{R}^3$ that tells us the current density at each location in space. Here is one way to understand the meaning of J: if S is a smooth surface in $\mathbb{R}^3$, then the rate at which current is flowing through S (in units of Coulombs per second) is $\int_S J \cdot dA$.
- A vector field B on $\mathbb{R}^3$ given by

$$B(x) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} J(y) \times \frac{(x - y)}{\|x - y\|^3} \, dy \tag{17}$$

  for all $x \in \mathbb{R}^3$. The constant $\mu_0 > 0$ is a parameter of the model. This formula for B is called the Biot-Savart formula.

- A vector F which represents the force on the charged particle due to the electric current. The vector F is assumed to satisfy

$$F = q(v \times B(x)) \tag{18}$$

  where $x \in \mathbb{R}^3$ is the location of the charged particle. Equation (18) is called the Lorentz force equation.

In this section we'll compute the divergence and the curl of this vector field B.

5.1 The divergence of B

Equation (17) can be written equivalently as

$$B(x) = \mu_0 \int_{\mathbb{R}^3} J(y) \times \nabla \Phi(x - y) \, dy$$

where Φ is the fundamental solution for the Laplace operator, defined in equation (13). Let $\epsilon > 0$ be a tiny positive number and define Φ_ϵ as in section 3.1. If $x \in \mathbb{R}^3$ then

$$B(x) \approx \mu_0 \int_{\mathbb{R}^3} J(y) \times \nabla \Phi_\epsilon(x - y) \, dy \tag{19}$$

and so

$$(\text{div } B)(x) \approx \mu_0 \int_{\mathbb{R}^3} \underset{\substack{\uparrow \\ \text{we are} \\ \text{taking the} \\ \text{divergence} \\ \text{with respect} \\ \text{to } x}}{\text{div}} \Big(J(y) \times \nabla\Phi_\epsilon(x-y) \Big)\, dy$$

$$= \mu_0 \int_{\mathbb{R}^3} \nabla\Phi_\epsilon(x-y) \cdot \Big(\underbrace{\nabla \times J(y)}_{0} \Big) - J(y) \cdot \Bigg(\underbrace{\nabla \times \nabla\Phi_\epsilon(x-y)}_{\substack{\text{curl of} \\ \text{gradient} \\ \text{is } 0}} \Bigg)\, dy$$

$$= 0$$

It seems plausible that we can make the approximation as close as we like by choosing ϵ to be sufficiently small. So we conclude that

$$\text{div } B = 0.$$

In the above calculation, we used the rule (7) to compute the divergence of the function

$$x \mapsto J(y) \times \nabla\Phi_\epsilon(x-y),$$

for any given $y \in \mathbb{R}^3$. The term $\nabla \times J(y)$ is equal to 0 because the curl is taken with respect to x.

5.2 Conservation of charge

Suppose that U is a bounded, open subset of $\mathbb{R}^3$ and that ∂U, the boundary of U, is a smooth surface. Since we are considering a *steady* current, electric charge must be entering the region U just as fast as charge is exiting. In other words, the rate at which charge is passing through ∂U is 0:

$$\int_{\partial U} J \cdot dA = 0.$$

Otherwise, we would have a violation of conservation of charge.

Applying the divergence theorem to the above equation, we see that

$$\int_U \text{div } J\, dx = 0.$$

Since U is arbitrary, it follows that

$$\text{div } J = 0. \tag{20}$$

Equation (20) expresses the conservation of charge.

5.3 The curl of B

Again, let $\epsilon > 0$ be a very small positive number. To simplify notation in the calculation below, let's define $\Psi = \nabla\Phi_\epsilon$. Starting from equation (19), we can take the curl of both sides and carefully apply the identity (8) to find that

$$\begin{aligned}(\nabla \times B)(x) &\approx \mu_0 \int_{\mathbb{R}^3} \nabla \times \Big(J(y) \times \Psi(x-y) \Big)\, dy \\ &\approx \mu_0 \int_{\mathbb{R}^3} -\Psi'(x-y)J(y) + (\text{div } \Psi)(x-y)J(y)\, dy \\ &= \underbrace{\mu_0 \int_{\mathbb{R}^3} -\Psi'(x-y)J(y)\, dy}_{\text{term 1}} + \underbrace{\mu_0 \int_{\mathbb{R}^3} \Delta\Phi_\epsilon(x-y)J(y)\, dy}_{\text{term 2}}\end{aligned}$$

Term 2 above is approximately $J(x)$, as can be seen by applying equation (10), which is the key property of a spike function. We'll show below that term 1 is 0. It follows that $\nabla \times B \approx \mu_0 J$. It seems plausible that we can make the approximation as close as we like by choosing ϵ to be sufficiently small. So, we conclude that

$$\nabla \times B = \mu_0 J.$$

To complete the argument, we must show that term 1 is 0. Let Ψ_1, Ψ_2, and Ψ_3 be the component functions of Ψ. I'll use the notation ∇_y to indicate the gradient taken with respect to y. The first component of term 1 is

$$\begin{aligned}\int_{\mathbb{R}^3} -\nabla\Psi_1(x-y) \cdot J(y)\, dy &= \int_{\mathbb{R}^3} \nabla_y \big(\Psi_1(x-y)\big) \cdot J(y) \\ &= -\int_{\mathbb{R}^3} \Psi_1(x-y) \underbrace{\text{div } J(y)}_{\substack{0 \\ \text{due to} \\ \text{conservation} \\ \text{of charge}}} \, dy \\ &= 0.\end{aligned}$$

We used the integration by parts formula (3) in the penultimate step. The boundary term in equation (3) vanishes here if we assume that $J(y)$ is 0 for sufficiently large y, which is a physically reasonable assumption. The electric current we are imagining is contained in some bounded region of space, and does not go "off to infinity" (3).

This shows that the first component of term 1 is 0. A similar calculation shows that the other components of term 1 are 0.