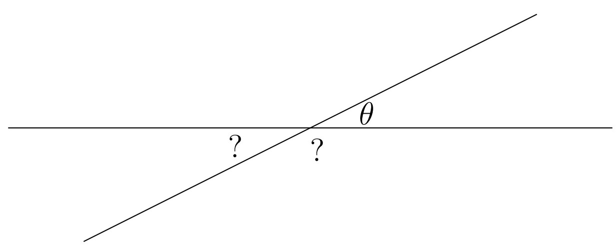
Problem Set 1
Summer Zero Math

It's ok to be concise, but you should explain your answers clearly, so that someone who does not already understand the answer could read your solution and understand it. When explaining math, you should usually write in full sentences, with correct punctuation and grammar.

1 The sum of the interior angles of a triangle

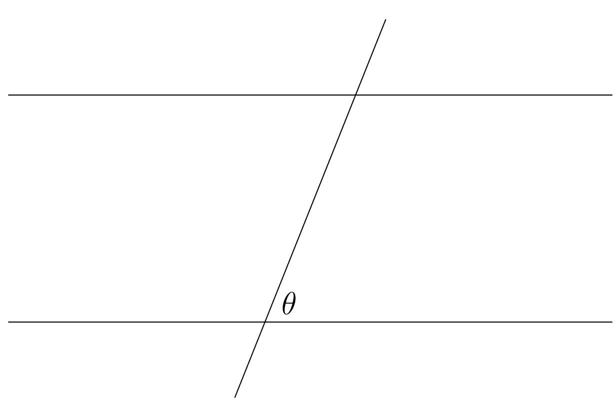
In this problem, assume that angles are measured using degrees.

a) Express the measures of the angles indicated below in terms of θ.

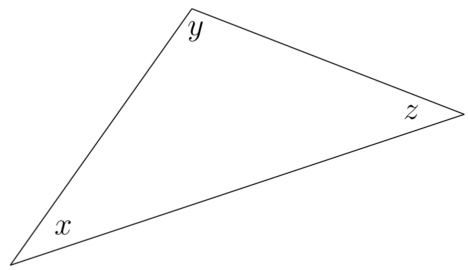


Fill in the blank: vertical angles are ________.

b) Express the measures of the unlabeled angles below in terms of θ. The two horizontal lines are parallel.



c) Suppose that the interior angles of a triangle have measures x, y, and z. What is $x + y + z$? Prove that your answer is correct.



2 The Pythagorean theorem

1. State the Pythagorean theorem.

2. You should know these examples by heart:

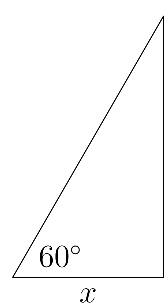
   a) The legs of a right triangle have length 3 and 4. What is the hypotenuse of the triangle?

   b) The legs of a right triangle have length 6 and 8. What is the hypotenuse of the triangle?

   c) The legs of a right triangle have length 5 and 12. What is the hypotenuse of the triangle?

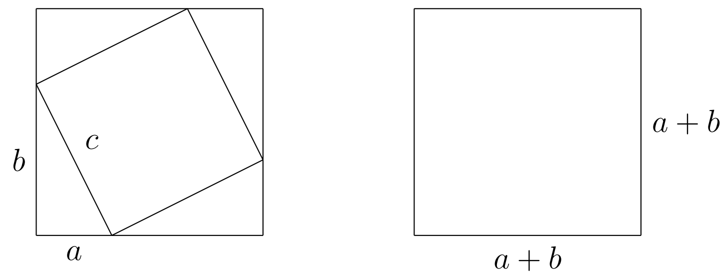
   d) The legs of a right triangle both have length x. What is the length of the hypotenuse of the triangle?

   e) Express the lengths of the other sides of this right triangle in terms of x:



3. Provide a proof of the Pythagorean theorem.

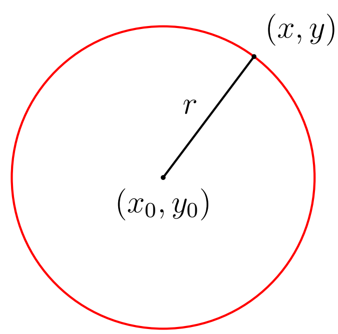
4. In the figure below on the left, a square with side $a+b$ has been dissected into four right triangles (with legs a, b and hypotenuse c) and a square with side c. Can you dissect the square on the right into four right triangles plus a square with side a and a square with side b? Does this provide an alternative proof of the Pythagorean theorem?



5. The sides of an equilateral triangle have length x. What is the height of the triangle?

6. (**The distance formula**) In a plane, point A has coordinates (x_1, y_1) and point B has coordinates (x_2, y_2). What is the distance from point A to point B?

7. A point (x, y) belongs to a circle of radius r centered at the point (x_0, y_0). Write down an equation that (x, y) satisfies, and explain why the equation is satisfied.



8. A point A in space has coordinates (x, y, z). What is the distance from the origin to A? Explain why your answer is correct.

9. (**Pythagorean triples**) An ordered triple (a, b, c) of positive integers satisfying

$$c^2 = a^2 + b^2$$

is called a "Pythagorean triple".

   (a) Give two examples of a Pythagorean triple.

   (b) Do you think there are infinitely many Pythagorean triples?

(c) Suppose that m and n are positive integers and $m > n$. Let $a = m^2 - n^2, b = 2mn, c = m^2 + n^2$. Find an interesting relationship between a, b, and c. What does this relationship tell you about whether or not there are infinitely many Pythagorean triples? (This was discovered by Euclid, the great ancient Greek mathematician who developed what we now call Euclidean geometry. Proving that there are infinitely many Pythagorean triples is a good example of the type of question studied in "number theory".)

Historical note: Let n be a positive integer. Let's call an ordered triple (a, b, c) of positive integers which satisfies

$$c^n = a^n + b^n$$

a "Fermat triple" with exponent n. The mathematician Fermat claimed (in the year 1637 or so) to have a wonderful proof that there are no Fermat triples with $n \geq 3$ "but the proof is too long to fit in the margin". This fact became known as "Fermat's last theorem". Despite intense efforts, Fermat's Last Theorem was not proved until 1995, over 350 years later. It was proved by Andrew Wiles, who spent 6 years working on the problem in secret before announcing his result. (It turned out his proof had a mistake, which took another year to fix, with the help of his past student Richard Taylor.)