Problem Set 2
Summer Zero Math

It's ok to be concise, but you should explain your answers clearly, so that someone who does not already understand the answer could read your solution and understand it. When explaining math, you should usually write in full sentences, with correct punctuation and grammar.

1 Using the "complete the square" trick to solve quadratic equations

The purpose of this section is to understand how the "completing the square" trick can be used to solve quadratic equations and to derive the quadratic formula.

1. This question concerns the meaning of the notation $\sqrt{w}$ (where w is a positive number).

   a) What are the possible values of x such that $x^2 = 25$?

   b) Is it correct to say that $\sqrt{25} = \pm 5$? Why or why not?

2. Suppose that x is a number.

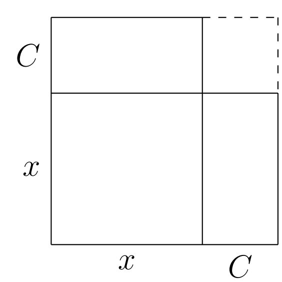
   (a) Expand $(x+2)^2$?

   (b) Expand $(x-5)^2$?

   (c) Expand $(x+C)^2$? (Here C is a number.)

   (d) What needs to be added to $x^2 + 2Cx$ in order to "complete the square"?

   (e) Explain how the picture below illustrates the answer to the previous question.



This picture shows that we are literally "completing" a square.

(f) What needs to be added to $x^2 + 6x$ in order to complete the square?

(g) What needs to be added to $x^2 + 5x$ in order to complete the square?

(h) What needs to be added to $x^2 + \frac{b}{a}x$ in order to complete the square? (Here a and b are numbers and $a \neq 0$.)

3. This problem attempts to illustrate how the "completing the square" trick reduces a hard problem to an easier problem. We start with an easy problem, and work our way up to a problem which at first glance appears more difficult.

   Solve for x:

   a) $(x + 2)^2 = 10$ (Easy, right?)

   b) $x^2 + 4x + 4 = 10$ (Why is this one easy?)

   c) $x^2 + 4x = 100$ (What's the trick to make this easy?)

   d) $x^2 + 4x - 50 = 0$ (How do you make this one easy?)

4. This problem attempts to illustrate how the "completing the square" trick reduces a hard problem to an easier problem. We start with an easy problem, and work our way up to a problem which at first glance appears more difficult.

   Solve for x:

   (a) $(x - 3)^2 = 50$.

   (b) $x^2 - 6x + 9 = 50$

   (c) $x^2 - 6x = 100$

   (d) $x^2 - 6x - 200 = 0$

   Note that part (a) is easy but part (d), at first glance, looks difficult. By "completing the square", part (d) is revealed to be no more difficult than part (a).

5. Complete the square and solve for x:

   a) $x^2 + 6x - 10 = 0$

   b) $x^2 + bx + c = 0$ (Here b and c are numbers.)

   Compare the formula for x that you get with the quadratic formula.

6. We now add one final wrinkle. Complete the square and solve for x:

   a) $5x^2 + 30x + 20 = 0$

   b) $ax^2 + bx + c = 0$ (Here a, b, and c are numbers and $a \neq 0$.)

   Boom! You just derived the quadratic formula.

c) In the previous question, why did we need the assumption that $a \neq 0$? What would be the formula for x if $a = 0$ but $b \neq 0$?

7. a) Is there a real number x which satisfies

$$x^2 - 2x + 8 = 0 \quad ? \tag{1}$$

b) What goes wrong when we attempt to solve equation (1) using the quadratic formula?

c) A very daring idea is to introduce a new number i which satisfies $i^2 = -1$. Introducing such a number might seem crazy and invalid at first, but it turns out to be a beautiful and extremely fruitful idea. So called "complex numbers" of the form $a + bi$ (where a and b are real numbers) are ubiquitous in higher mathematics, and for example feature prominently in quantum mechanics and in the math that underlies magnetic resonance imaging. (MRI scans, performed every day at hospitals all over the world, rely on complex numbers.)

Using this new number i which satisfies $i^2 = -1$, write down two solutions to problem (1).

Optional note: If you want a precise definition of a complex number, you can define a complex number to be an ordered pair (a, b) of real numbers. So the complex number that we would usually write as $a + bi$ is in fact, by definition, just the ordered pair (a, b). You can visualize $a + bi$ as a point in a plane with coordinates (a, b). With this definition, $i = 0 + 1 \cdot i$ is, by definition, the ordered pair $(0, 1)$. A complex number such as -1, whose imaginary part is 0, is by definition the ordered pair $(-1, 0)$. This way of defining complex numbers avoids any objection that there is "no such thing" as a number i which satisfies $i^2 = -1$. In this approach to developing the complex number system, we must also define addition and multiplication of complex numbers. It seems natural to define

$$(a, b) + (c, d) = (a + c, b + d).$$

Usually we would write this as

$$(a + bi) + (c + di) = (a + b) + (c + d)i.$$

How to define multiplication of complex numbers is less obvious, but we should do it in such a way that the usual rules of arithmetic remain true. For example, the distributive rule should remain true. It should also be true that $i^2 = -1$, or in other words $(0, 1) \times (0, 1) = (-1, 0)$. It's a good puzzle to see what definition of multiplication of complex numbers these restrictions lead to:

$$(a, b) \times (c, d) = ?$$

In other words:

$$(a + bi)(c + di) = ?$$

Historical note (from Wikipedia):

We know how to solve quadratic equations

$$ax^2 + bx + c = 0$$

but what about solving cubic equations

$$ax^3 + bx^2 + cx + d = 0$$

or quartic equations

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

or quintic equations

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \quad ?$$

- In the early 16th century, the Italian mathematician Scipione del Ferro (1465–1526) found a method for solving a class of cubic equations, namely those of the form $x^3 + mx = n$. In fact, all cubic equations can be reduced to this form if we allow m and n to be negative, but negative numbers were not known to him at that time. Del Ferro kept his achievement secret until just before his death, when he told his student Antonio Fior about it.

  In 1530, Niccolò Tartaglia (1500–1557) received two problems in cubic equations from Zuanne da Coi and announced that he could solve them. He was soon challenged by Fior, which led to a famous contest between the two. Each contestant had to put up a certain amount of money and to propose a number of problems for his rival to solve. Whoever solved more problems within 30 days would get all the money. Tartaglia received questions in the form $x^3 + mx = n$, for which he had worked out a general method. Fior received questions in the form $x^3 + mx^2 = n$, which proved to be too difficult for him to solve, and Tartaglia won the contest.

  Later, Tartaglia was persuaded by Gerolamo Cardano (1501–1576) to reveal his secret for solving cubic equations. In 1539, Tartaglia did so only on the condition that Cardano would never reveal it and that if he did write a book about cubics, he would give Tartaglia time to publish. Some years later, Cardano learned about del Ferro's prior work and published del Ferro's method in his book Ars Magna in 1545, meaning Cardano gave Tartaglia six years to publish his results (with credit given to Tartaglia for an independent solution). Cardano's promise to Tartaglia said that he would not publish Tartaglia's work, and Cardano felt he was publishing del Ferro's, so as to get around the promise. Nevertheless, this led to a challenge to Cardano from Tartaglia, which Cardano denied. The challenge was eventually accepted by Cardano's student Lodovico Ferrari (1522–1565). Ferrari did better than Tartaglia in the competition, and Tartaglia lost both his prestige and his income.

Cardano noticed that Tartaglia's method sometimes required him to extract the square root of a negative number . He even included a calculation with these complex numbers in Ars Magna, but he did not really understand it. Rafael Bombelli studied this issue in detail and is therefore often considered as the discoverer of complex numbers.

- Lodovico Ferrari is attributed with the discovery of the solution to the quartic in 1540, but since this solution, like all algebraic solutions of the quartic, requires the solution of a cubic to be found, it couldn't be published immediately. The solution of the quartic was published together with that of the cubic by Ferrari's mentor Gerolamo Cardano in the book Ars Magna (1545).
- Amazingly, the French mathematician Galois proved that there is no quintic formula (and no formula for higher degree polynomial equations) prior to dying in a dual in 1832, at the age of 20. Despite living only to the age of 20, Galois was one of the greatest mathematicians of all time. An important subject in pure math called Galois theory is based on his work.