Problem Set 3
Summer Zero Math

It's ok to be concise, but you should explain your answers clearly, so that someone who does not already understand the answer could read your solution and understand it. When explaining math, you should usually write in full sentences, with correct punctuation and grammar.

1 The story of exponent notation

Let a be a positive real number.

1. If we didn't have exponent notation, how would you have to write a^{10}?

   Clearly this simple exponent notation already saves a lot of writing, and allows us to write equations more concisely and clearly. What's surprising is how far we can push or extend this notation to make it ever more expressive and useful. When exponent notation is first introduced, the exponent is assumed to be a positive integer. But then we discover that the notation can be fruitfully extended so that the exponent is allowed to be 0, or a negative integer, or a fraction (a rational number), or any real number. In more advanced math, the notation is extended further still so that the exponent is allowed to be a complex number (!) or even a matrix.

2. Explain why $a^7 \cdot a^{10} = a^{17}$.

3. Suppose that m and n are positive integers. Then

$$a^m \cdot a^n = a^? \tag{1}$$

   Explain why this formula is true.

   We love this simple formula (1) because it makes calculations involving exponents very easy. If we are going to define a^x where x is something other than a positive integer, we certainly want to come up with a definition such that the rule (1) remains true.

4. Explain why

$$(a^5)^3 = a^{15}.$$

5. Suppose that m and n are positive integers. Then

$$(a^m)^n = a^? \tag{2}$$

Explain why this rule is true.

We also love the simple formula (2). When we attempt to define a^x for other types of numbers x, we certainly hope that the rule (2) will remain true.

6. Now we will decide how to define a^0. According to rule (1) , what should $a^1 \cdot a^0$ be equal to?

$$a^1 \cdot a^0 = a^?$$

So, how must we define a^0 if our goal is to preserve rule (1)?

$$a^0 = ?$$

7. Suppose that n is a positive integer. How must we define a^{-n} if rule (1) is to remain true?

$$a^{-n} = ?$$

8. How must we define $a^{1/2}$ in order for rule (2) to remain true?

$$a^{1/2} = ?$$

9. Suppose that m and n are positive integers. How must we define $a^{m/n}$ in order for rule (2) to remain true?

$$a^{m/n} = ? \tag{3}$$

10. You might worry that the definition (3) we have given of $a^{m/n}$ is ambiguous, because there are many ways to represent the same fraction. For example, $2/3 = 4/6$. When evaluating $a^{2/3}$ using definition (3), does it matter if we take $m = 4$ and $n = 6$ instead of $m = 2$ and $n = 3$? Or do we get the same result either way. Explain your reasoning.

11. If m and n are positive integers, how must $a^{-m/n}$ be defined in order for rule (1) to remain true?

12. A "rational number" is a number x which has the form

$$x = \frac{m}{n}$$

where m and n are integers. We have now defined a^x for any rational number x. However, some numbers such as $\pi = 3.14159...$ are not rational. In fact, in some sense most real numbers are not rational. One way to understand this intuitively is as follows: it can be shown that a real number x is rational if and only if the decimal expansion of x eventually repeats. But for a decimal expansion to eventually repeat is a very special, rare event. If you were generating the decimal digits of a number x at random, most likely the decimal expansion you generate would never begin to repeat (due to the randomness).

An "irrational" number is a real number which is not rational. Our next step in the story of exponent notation is to define a^x in the case where x is an irrational number.

How would you define a^π, for example?

Note: We must wait until Calculus in order to define a^x when x is a complex number. But when we do, we'll discover "Euler's identity", one of the most beautiful formulas in math.

2 Logarithms

In this section, assume that $b > 1$ and $x, y > 0$.

1. How small can b^x possibly be (assuming that x is a real number)? How would you choose x to make 2^x very small?

2. Is it possible to choose x so that b^x is negative? What about $b^x = 0$, is that possible?

3. In high school algebra, we solve equations such as $2x + 8 = 40$. But what if the unknown number x were an exponent? For example, can you solve

$$2^x + 8 = 40 \qquad ?$$

   In this section we introduce "logarithm" notation which is helpful for solving equations like this, where the unknown appears in an exponent.

4. Solve the following equations for x:

   a) $2^x = 16$
   b) $2^x = \frac{1}{16}$
   c) $2^x = 1$
   d) $3^x = 27$
   e) $3^x = \frac{1}{27}$
   f) $3^x = 1$
   g) $10^x = 100$
   h) $10^x = \frac{1}{100}$
   i) $10^x = 1$

5. If y is a positive number, there is a unique number x such that

$$b^x = y.$$

This number x has been given the unfortunate name "the logarithm base b of y", and it is denoted $\log_b(y)$. A much more clear name would be "the exponent from b to y". The whole point of this number $\log_b(y)$ is that it is the exponent which satisfies

$$b^? = y.$$

In other words,

$$b^{\log_b(y)} = y. \tag{4}$$

I think a more clear notation for $\log_b(y)$ would be $[b \to y]$. With this notation, equation (4) would be written

$$b^{[b\to y]} = y$$

which might seem more readable. I'll call this the "alternative" notation for logarithms. (Note that I made this notation up and nobody else uses it.)

Evaluate the following:

a) $\log_2(64)$

b) $\log_{10}(1000)$

c) $\log_3(1/9)$

d) $\log_5(1/125)$

e) $\log_5(1)$

In the next several problems we are going to understand why the following fundamental properties of logarithms are true:

A. $\log_b(b^x) = x$ for any real number x.

B. $\log_b(1) = 0$ and $\log_b(b) = 1$.

C. (**logarithm of product**) $\log_b(y_1 y_2) = \log_b(y_1)+\log_b(y_2)$ for any positive numbers y_1, y_2. (In words, the logarithm of the product is the sum of the logarithms.)

D. (**logarithm of quotient**) $\log_b(y_1/y_2) = \log_b(y_1) - \log_b(y_2)$ for any positive numbers y_1, y_2. (In words, the logarithm of the quotient is the difference of the logarithms.)

E. (**logarithm of a power**) $\log_b(y^w) = w \log_b(y)$ for any real numbers y, w with $y > 0$. (Informally, you can pull the exponent out front.)

F. (**change of base**) If $a > 1$ then $\log_a(y) = \frac{\log_b(y)}{\log_b(a)}$ for any positive number y.

6. Explain why rule A. is true.

7. Explain why rule B. is true.

8. Suppose that y_1 and y_2 are positive numbers. Let $x_1 = \log_b(y_1)$ and $x_2 = \log_b(y_2)$. This means that

$$b^{x_1} = y_1 \quad \text{and} \quad b^{x_2} = y_2.$$

   a) Can you find an exponent such that

   $$b^? = y_1 y_2 \quad ?$$

   So, what is $\log_b(y_1 y_2)$? Explain why rule C. is true.

   b) Can you find an exponent such that

   $$b^? = \frac{y_1}{y_2} \qquad ?$$

   So, what is $\log_b(y_1/y_2)$? Explain why rule D. is true.

   c) Suppose that y and w are real numbers and that y is positive. Let $x = \log_b(y)$. Can you find an exponent such that

   $$b^? = y^w \qquad ?$$

   So, what is $\log_b(y^w)$? Explain why rule E. is true.

   d) The change of base formula given in rule F. can be written equivalently as

   $$\log_b(y) = \log_b(a)\log_a(y). \tag{5}$$

      a) How do you get equation (5) from the formula in rule F.?
      b) Rewrite equation (5) using the "alternative" notation for logarithms. How might you express this equation in words?
      c) Explain why formula (5) is true.

   e) What does the graph of the function

   $$f(x) = 2^x \tag{6}$$

   look like?

   This function f increases very rapidly. Explain why, if the input to f increases by 1, then the output of f is doubled. This function f could be useful for describing a growing population which doubles in size every year, for example. Write down a function which describes the growth of a population which doubles every 60 years, and which initially has 1000 members.

f) What does the graph of the function $g(y) = \log_2(y)$ look like? This function g increases very slowly. Explain why, if the input to g increases by a factor of 2, the output of g increases only by 1. (For example, if the input increases from 1000 to 2000, the output increases by only 1.)

This function g is called the "inverse" of the function f given in equation (6), for the following reason: g takes the output of f and sends it back where it came from.

- Notice that $f(5) = 32$. What is $g(32)$?
- Notice that $f(10) = 1024$. What is $g(1024)$?
- Explain why $g(f(x)) = x$ for any real number x.