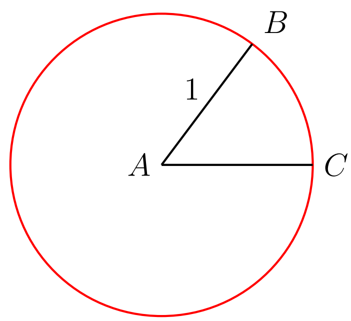
Problem Set 4
Summer Zero Math

It's ok to be concise, but you should explain your answers clearly, so that someone who does not already understand the answer could read your solution and understand it. When explaining math, you should usually write in full sentences, with correct punctuation and grammar.

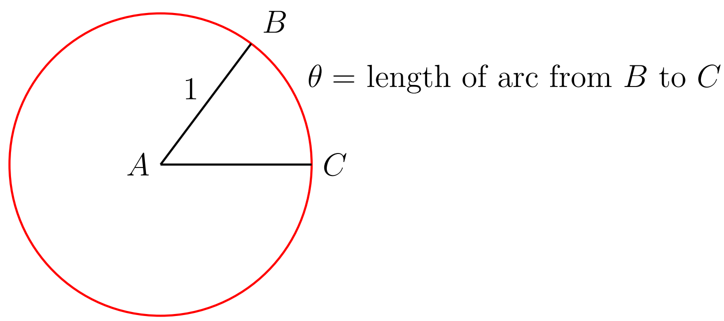
1 Trigonometry

1. In math, we need a convenient way to describe the size of an angle, such as the angle $\angle BAC$ below which is inscribed in a circle of radius 1.



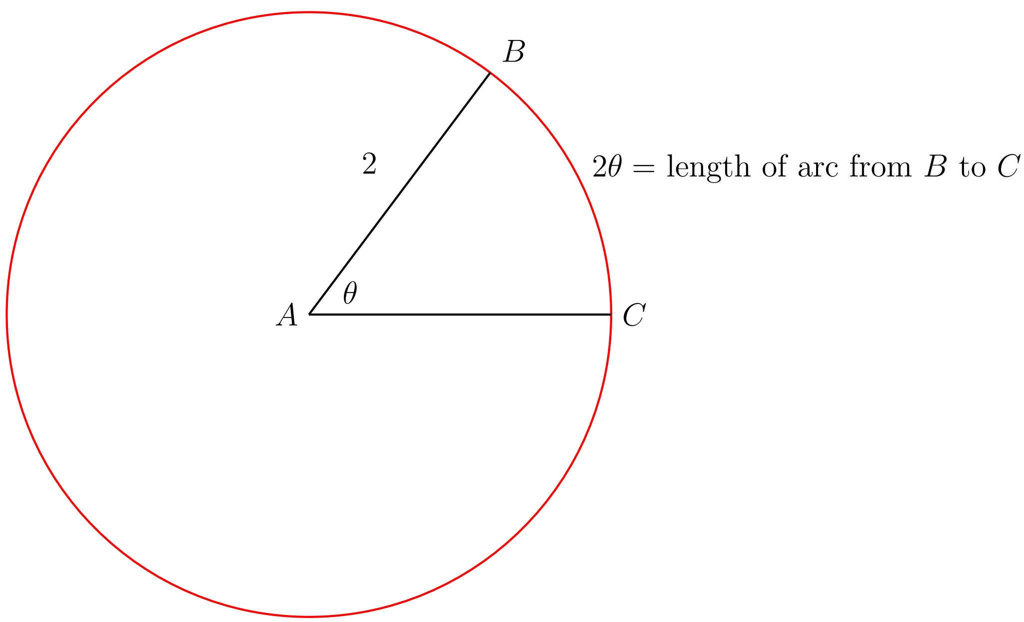
Often people measure the size of an angle using degrees, but, the degree system is somewhat arbitrary. Why are there 360 degrees in a circle? This was merely an arbitrary choice made by humans. This arbitrariness is aesthetically displeasing, and leads to more complicated formulas for things like arc length, the area of a sector of a circle, etc. We want our math to be as simple and elegant as possible.

A different and more natural way to measure the size of the angle $\angle BAC$ is to simply measure the length of the arc of the circle from B to C. This number is called the "radian measure" of the angle $\angle BAC$.



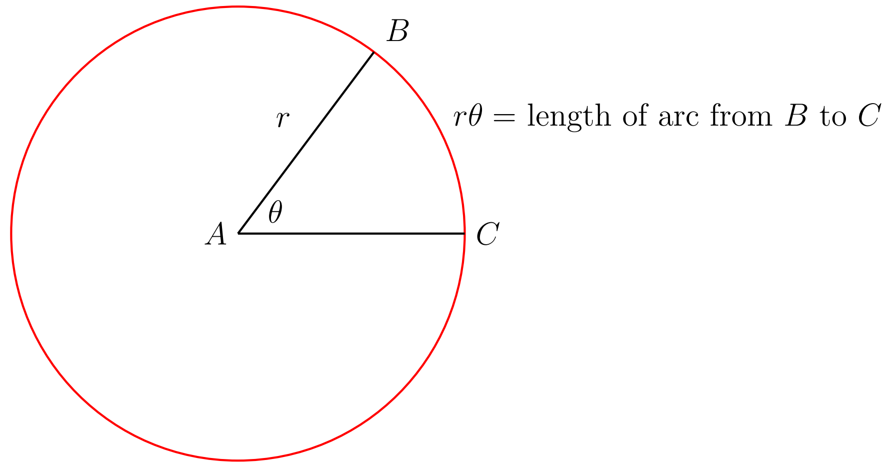
The radian measure of $\angle BAC$ is the **arc length** θ.

Notice that when we define the radian measure of an angle, we imagine that the angle is inscribed in a circle of radius 1. If the circle were doubled in size, then the new radius of the circle would be 2 and the new arc length from B to C would be 2θ, but the measure of the angle would not change. This is illustrated below.



In this picture, the original circle has been doubled in size. The radian measure of $\angle BAC$ is still θ, but now the arc length from B to C is 2θ .

More generally, if the original circle were scaled by a factor of r (where $r > 0$), then the new radius of the circle would be r and the new arc length from B to C would be $r\theta$, as illustrated below.



In this picture, the original circle has been scaled by a factor of r. The radian measure of $\angle BAC$ is still θ, but now the arc length from B to C is $r\theta$.

The arc length formula in the above picture is worth emphasizing:

$$\text{length of arc from } B \text{ to } C = r\theta \tag{1}$$

where θ is the radian measure of the angle $\angle BAC$.

a) How many radians are there in a full circle? In half a circle? In a quarter of a circle? In an eighth of a circle?

b) Express your answer to the previous question in terms of the number $\tau = 2\pi$. (Some people think that mathematical formulas tend to be more elegant when written in terms of τ rather than π.)

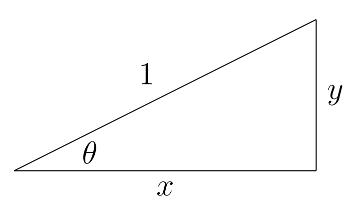
c) How many degrees are in 1 radian?

d) How many radians are in 1 degree?

e) Convert $\pi/4$ radians to degrees.

f) Convert 30 degrees to radians.

2. Here is a very simple and fundamental question which is not easy to answer. The right triangle shown below has hypotenuse 1. If someone tells us the value of θ (in radians), how do we compute x and y?



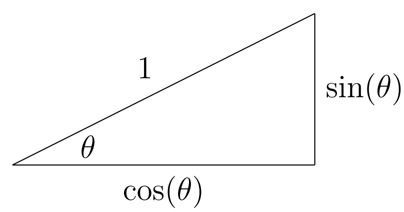
One might think this problem is easy, because by definition x is the "cosine" of θ and y is the "sine" of θ:

$$x = \cos(\theta), \qquad y = \sin(\theta).$$

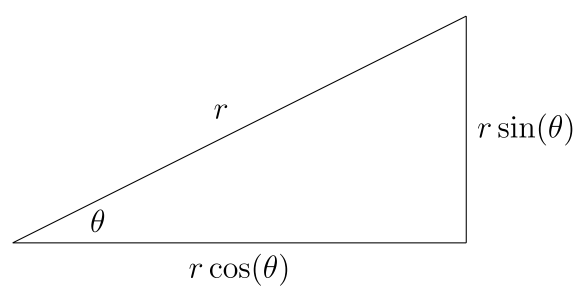
However, all we have done so far is given new names to the unknowns x and y. We have not said anything about how to compute them! Giving a new name to the unknown is not the same as explaining how to calculate the value of the unknown.

One might say: well, just ask the calculator. But how does the calculator do it? That's the whole question.

Introducing names for the unknowns is a good first step, though. Now we can talk about these sine and cosine functions more easily. If we can understand these functions well enough, we might eventually figure out how to compute their values. Here is the above picture again, using our new names for x and y:



If the triangle above is scaled by a factor of r then the new triangle we get looks like this:



When we scale the triangle, the length of each side changes (by a factor of r), but the ratios of the lengths of various sides do not change:

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{r\sin(\theta)}{r} = \sin(\theta), \tag{2}$$

$$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{r\cos(\theta)}{r} = \cos(\theta), \tag{3}$$

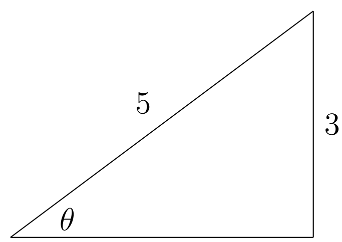
$$\frac{\text{opposite}}{\text{adjacent}} = \frac{r\sin(\theta)}{r\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)}. \tag{4}$$

This final ratio is called the "tangent" of θ:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{opposite}}{\text{adjacent}}.$$

The mnemonic "Soh Cah Toa" is helpful for remembering the formulas (2)-(4).

a) What are $\sin(\theta)$ and $\cos(\theta)$ for the angle θ in the following picture? (This triangle is a right triangle.)



b) What are the sine, cosine, and tangent of $\theta = \pi/4$? Explain your answer.

c) What are the sine, cosine, and tangent of $\theta = \pi/3$? Explain your answer.

d) Suppose that $0 < \theta < \pi/2$. What is $\sin^2(\theta) + \cos^2(\theta)$ equal to?

$$\sin^2(\theta) + \cos^2(\theta) = \quad ?$$

Explain your answer. (The notation $\sin^2(\theta)$ is a short way of writing $(\sin(\theta))^2$.)

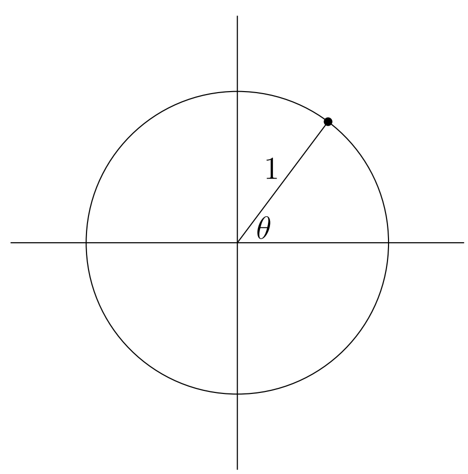
e) Suppose that $0 < \theta < \pi/2$. Explain why

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$$

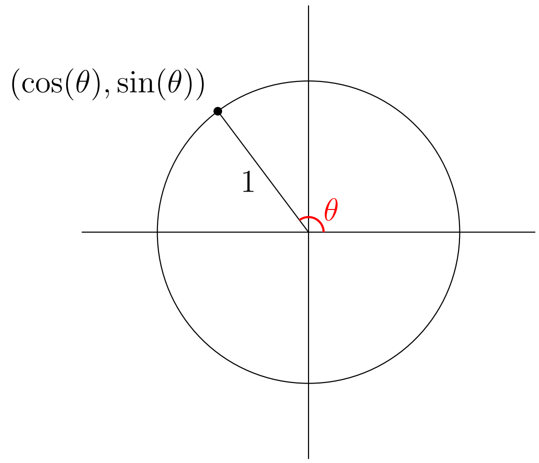
and

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right).$$

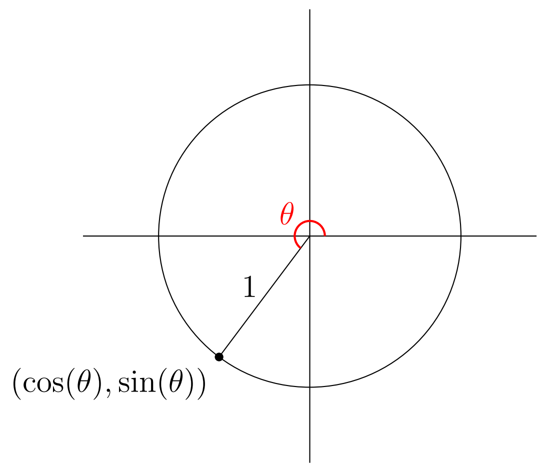
f) What are the coordinates of this point on the unit circle:



3. When we introduced sine and cosine, we implicitly assumed that θ was between 0 and $\pi/2$. But, the above picture suggests a way to define $\cos(\theta)$ and $\sin(\theta)$ even when θ is greater than $\pi/2$ or less than 0. This is best illustrated with pictures, shown below.



We can **define** $\cos(\theta)$ and $\sin(\theta)$ to be the coordinates of the corresponding point on the unit circle.



In words: imagine starting at the point $(1, 0)$ and moving counterclockwise along the unit circle until we have swept out θ radians. We now **define** $\cos(\theta)$ and $\sin(\theta)$ to be the coordinates of the point where we arrive on the unit circle.

a) Complete the following table:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos(\theta)$									
$\sin(\theta)$									

b) Explain why $\sin(-\theta) = -\sin(\theta)$.

c) Explain why $\cos(-\theta) = \cos(\theta)$.

d) Simplify $\sin(\theta + \pi)$.

e) Simplify $\cos(\theta + \pi)$.

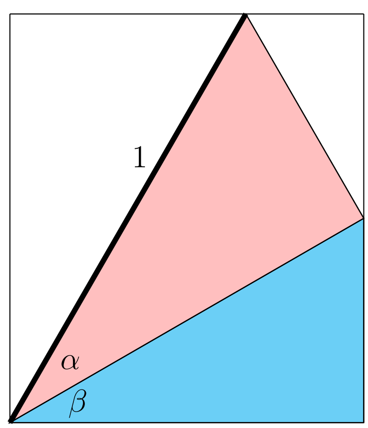
f) Simplify $\sin(\theta + \pi/2)$.

g) Simplify $\cos(\theta + \pi/2)$.

h) What are the periods of the sine and cosine functions?

i) Graph sine and cosine on the same axes.

4. State and prove the addition formulas for sine and cosine. The picture below will be helpful:



5. Using the addition formulas for sine and cosine to derive formulas for the following:

a) $\sin(x - y)$

b) $\cos(x - y)$

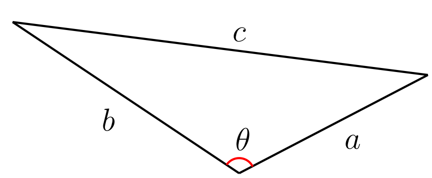
c) $\sin(2x)$

d) $\cos(2x)$

e) $\sin(3x)$

f) $\cos(3x)$

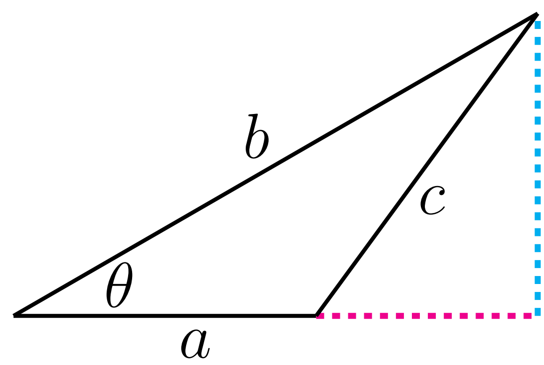
6. The "Law of Cosines" is a generalization of the Pythagorean theorem that applies to non-right triangles.



Law of Cosines: Suppose that two sides of a triangle have lengths a and b, and the angle between these two sides is θ. Let c be the length of the third side of the triangle. Then

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

a) Prove the Law of Cosines. The picture below might help:



b) Why is the Pythagorean theorem a special case of the Law of Cosines?