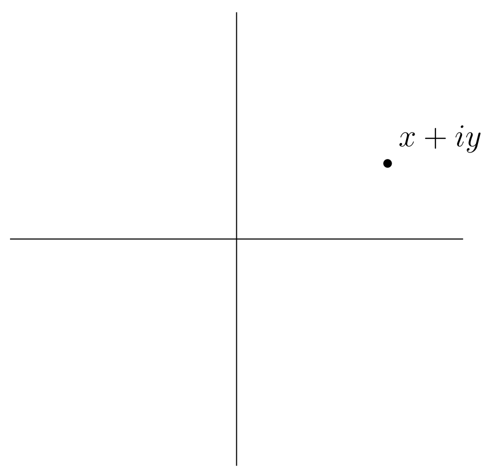
Problem Set
Summer Zero Math

It's ok to be concise, but you should explain your answers clearly, so that someone who does not already understand the answer could read your solution and understand it. When explaining math, you should usually write in full sentences, with correct punctuation and grammar.

1 Complex numbers

In this section we explore the **visual interpretation** of complex numbers.

1. You can visualize a complex number $x + iy$ (where x and y are real numbers and $i^2 = -1$) by plotting a point in a plane with coordinates (x, y). This is illustrated below.



The "norm" of a complex number is denoted $\|x + iy\|$ and is defined as follows:

$$\|x + iy\| = \sqrt{x^2 + y^2}.$$

a) Draw a picture of the complex number $3 + 4i$ and compute its norm.

b) Draw a picture of the complex number $-6 + 8i$ and compute its norm.

c) Draw a picture of the complex number i and compute its norm.

d) Draw a picture of the complex number $-i$ and compute its norm.

e) Can you give a geometric interpretation of the norm of a complex number $x+iy$? Explain your answer.

2. Compute the following and illustrate with a picture:

   (a) $(1+i)+(2+3i)$

   (b) $(3+2i)+(3-2i)$

   (c) $(2+i)+(-4+2i)$

   Can you give a geometric interpretation of addition of complex numbers?

3. Compute the following:

   a) $(2+i)(1+3i)$

   b) $(1+i)(1-i)$

   c) $(1+i)^2$

   d) $i(3+4i)$

4. Compute the norm of a, the norm of b, and the norm of ab, when

   a) $a = 3+4i, b = 5+12i$

   b) $a = i+i, b = 1-i$

   Can you make a conjecture about the norm of the product of two complex numbers?

5. Compute the angle that a makes with the real axis, the angle that b makes with the real axis, and the angle that ab makes with the real axis, when

   a) $a = 1+i, b = 1+i$

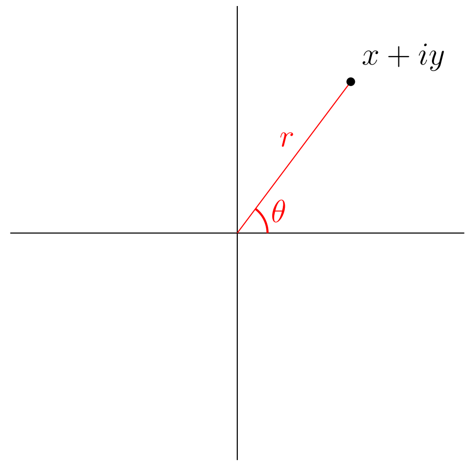
   b) $a = 1+i, b = -1+i$

   c) $a = 1+i, b = -1-i$

   d) $a = 1+\sqrt{3}i, b = 1+\sqrt{3}i$

   Based on these results, can you make a conjecture about what happens visually when you multiply two complex numbers?

6. Express the complex number $x+iy$ (shown below) in terms of r and θ.



This way of writing $x + iy$ is called the "polar form" of $x + iy$.

7. Write the following complex numbers in polar form:

   a) $1 + i$
   b) $1 + \sqrt{3}i$
   c) $1 - i$
   d) $-1 + \sqrt{3}i$

8. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers. (Here x_1, y_1, x_2, and y_2 are real numbers.) Express both z_1 and z_2 in polar form, then compute the product of z_1 and z_2.

$$z_1 z_2 = \qquad ?$$

   Key step: Simplify the result. (Aha!)

   What did you discover? Provide a visual interpretation of complex number multiplication. (This is one of the key results of Summer Zero Math.)

9. Let θ be a positive number.

   a) What is $(\cos(\theta) + i\sin(\theta))^2$?
   b) What is $(\cos(\theta) + i\sin(\theta))^3$?
   c) If n is a positive integer, what is $(\cos(\theta) + i\sin(\theta))^n$?

10. How many solutions can you find to the equation

$$z^6 = 1$$

   where z is a complex number? Draw a picture of the solutions you find.

If n is a positive integer, how many solutions are there to

$$z^n = 1$$

where z is a complex number? Draw a picture. Complex numbers z that satisfy $z^n = 1$ are called "nth roots of unity".