Problem Set 6
Summer Zero Math

It's ok to be concise, but you should explain your answers clearly, so that someone who does not already understand the answer could read your solution and understand it. When explaining math, you should usually write in full sentences, with correct punctuation and grammar.

1 Solving linear systems of equations

1. Solve for x_1 and x_2 using the substitution method:

$$\begin{aligned} x_1 + 2x_2 &= 9, \\ 3x_1 + 5x_2 &= 20. \end{aligned}$$

2. Solve for x_1 and x_2:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1, \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

   What assumption do you need in order for your formula to be valid?

3. Solve for x_1, x_2, and x_3:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

   What assumption do you need in order for your formula to be valid?

4. (**Introducing matrix and vector notation**) There is something wasteful about the way the linear system above is written. The numbers x_1, x_2, and x_3 have each been written three times. That is a waste of ink. Isn't there a more concise way to express the same information?

   We now invent a more concise notation. In our new notation, the above linear system of equations will be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

   The 3×3 array of numbers on the left is called a "matrix". The vertical lists of numbers

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   are called "vectors" or "column vectors". As part of our new notation, we are making a declaration or an agreement that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{is a short way of writing} \quad \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{bmatrix}.$$

   By the way, if we name this matrix A and name these column vectors x and b, then our system of equations can be written extremely concisely as $Ax = b$:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}.$$

   It is tempting to think of the matrix A as a thing that performs an operation on the column vector x. We shall take this viewpoint. When we write Ax, we will say that we are "multiplying" the column vector x by the matrix A, and the result is a new column vector:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \text{“}A \text{ times } x\text{”} = \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}}_{\substack{\text{This column vector is the} \\ \text{result of “multiplying” } x \text{ by } A.}}.$$

   So we have now learned how to "multiply" a matrix by a vector. When computing Ax, it helps to notice that the ith entry of the result is the "sum-product" (also called "dot product") of the ith row of A with x.

   Write the following linear systems of equations in matrix notation:

a)

$$\begin{aligned} x_1 + 2x_2 &= 9, \\ 3x_1 + 5x_2 &= 20. \end{aligned}$$

b)

$$\begin{aligned} y - x &= 1, \\ z - y &= 2, \\ x - z &= -3. \end{aligned}$$

c)

$$\begin{aligned} x + y + z &= 3, \\ y + z &= 2, \\ z &= 1. \end{aligned}$$

d)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1, \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

e)

$$\begin{aligned} ax + by + cz &= u, \\ dx + ey + fz &= v. \end{aligned}$$

(The unknowns are x, y, and z.)

5. Compute the following:

   a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

   b) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$

6. Solve the following linear system of equations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

   What assumption do you need to make in order for your solution to be valid?

   Hint: This question is identical to a previous question, except now we're using matrix notation.

7. Let

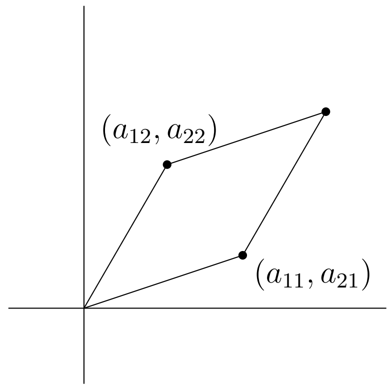
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The "determinant" of A is the number

$$\det A = a_{11}a_{22} - a_{21}a_{12}.$$

a) What is the significance of this number?

b) What is the area of the parallelogram shown below?



8. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be 2×2 matrices. Find a 2×2 matrix C such that

$$Cx = A(Bx)$$

for every vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. This matrix X is called the "product" of A and B. So we have now learned how to multiply a matrix by another matrix. (At least, we have discovered the matrix multiplication operation for 2×2 matrices. You can ponder how to extend this operation to matrices of other shapes.)

9. The matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is called the 2×2 identity matrix.

Compute the following:

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

What happens when you multiply a 2×2 matrix or a 2×1 column vector by I? Why do you think I is called an "identity matrix"?

10. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Find a 2×2 matrix M such that

$$MA = AM = I$$

where I is the 2×2 identity matrix.

What assumption do you need to make in order for M to exist? This matrix M, if it exists, is called the "inverse" of A.

11. a) Compute the inverse of the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

b) Use the result from part a) to solve

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

12. Humans love to visualize math, and the most obvious way to visualize an ordered list of numbers $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is to think of x, y, and z as being the coordinates of a point in space. You can visualize that point. In this way of visualizing v, which I call the "point picture", we think of the list of numbers $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as specifying a *location* in space.

Here is a different, but equally important, way to visualize an ordered triple $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.
We draw (or imagine) a 3D coordinate system, and we select a point P in space arbitrarily. Starting at P, we move a distance x in the direction of the x-axis, then we move a distance y in the direction of the y-axis, and then we move a distance z in the direction of the z-axis. We arrive at a point Q. To visualize our vector v, we draw (or imagine) an arrow that starts at P and ends at Q. In this picture, which I call the "vector picture", we think of v as specifying the *displacement* from the point P to the point Q.

Note that the starting point P is selected arbitrarily. If we select a different starting point, then we will draw a different arrow. But, the arrow will still have the same length and the same direction as the original arrow.

You can draw similar pictures to visualize an ordered pair $v = \begin{bmatrix} x \\ y \end{bmatrix}$ that has only two components rather than three. But in this case, you would start by drawing a 2D coordinate system rather than a 3D coordinate system.

When we visualize an ordered pair or an ordered triple v using the point picture, we call it a "point". When we visualize an ordered pair or an ordered triple v using the vector picture, we call it a "vector". Either way, v is simply an ordered list of numbers. The only difference is what we visualize when we think about v. (Warning: most people, including myself, are not perfectly consistent about this terminology. But I try to be.)

Draw pictures to visualize the following vectors. (Use the "vector picture" rather than the "point picture".)

a) $u = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b) $v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

c) $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

d) $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

e) $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

13. The length of the arrow that we draw when visualizing a vector v is called the "magnitude" or the "norm" of the vector, and is denoted $\|v\|$.

a) Compute the norm of the vector $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

b) Give a formula for the norm of the vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$.

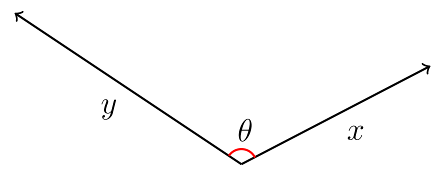
c) Give a formula for the norm of the vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

14. The "dot product" of vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is defined to be the number

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2.$$

Prove that if θ is the angle between the vectors x and y then

$$\langle x, y \rangle = \|x\| \|y\| \cos(\theta).$$



(Hint: use the Law of Cosines.)